Math 250 4.2 - 2nd What Derivatives Tell Us
Concavity and the Second Derivative

Objectives

1) Determine intervals on which a function is concave up or concave down
2) Find points of inflection
3) Apply the second derivative test to find relative extrema of function

M250 Intervals of Concavity and 2nd Derivative Test for Extrema

A function *f* is <u>concave upward (or concave up)</u> on (a,b) if $f''(x) > 0$ for all x in (a,b).
This means that f' is increasing, or, for any two numbers x_1 and x_2 in (a,b),
If $x_1 < x_2$ then $f'(x_1) < f'(x_2)$. The graph "smiles".

A function *f* is <u>concave downward (or concave down)</u> on (a,b) if $f''(x) < 0$ for all x in (a,b).
This means that f' is decreasing, or, for any two numbers x_1 and x_2 in (a,b),
If $x_1 < x_2$ then $f'(x_1) > f'(x_2)$. The graph "frowns".

The first derivative f' is <u>constant</u> on (a,b) if $f''(x) = 0$ for all x in (a,b).
This means that for any two numbers x_1 and x_2 in (a,b), $f'(x_1) = f'(x_2)$.
This also means that *f* is linear on (a,b).

Since $f''(x)$ gives the rate of change of the slope of the tangent line wherever it is defined on the graph of *f*,

Concave up at x_1 ⇔ slope increasing at x_1 ⇔ $f''(x_1) > 0$

Concave down at x_1 ⇔ slope decreasing at x_1 ⇔ $f''(x_1) < 0$

(Neither concave up nor concave down) Linear at x_1 ⇔ slope constant at x_1 ⇔ $f''(x_1) = 0$

<u>To determine intervals of concavity</u>:

1) Find the first and second derivatives.
2) Find the values of x where $f''(x) = 0$ and where $f''(x)$ is undefined.

{<u>Note</u>: These are not necessarily the same as the critical values, but sadly, there's no name for them.}

3) Graph the values from step 2 in numerical order on a number line.
4) Label the number line f''
5) Determine the sign of the second derivative at a test point for each interval.
6) Write open intervals:
   - positive f'' at the test point are concave up
   - negative f'' at the test point are concave down

<u>CAUTION</u>: When testing values, be sure to test using f'', not *f* or f'.
<u>Note</u>: Sometimes it is easier to look at the graph of f''. If the graph is above the x-axis, its value is positive. If the graph is below the x-axis, its value is negative.
<u>CAUTION</u>: The *concavity* of the graph of f'' is usually unrelated to the *concavity* of *f*.

<u>To determine if a critical value is an extremum using the second derivative test</u>:

1) Find the critical values:

<u>Note</u>: The critical values are *always* $f' = 0$ or f' undefined.

2) Evaluate f'' for each critical value.
   - If $f'' = 0$, the second derivative test is inconclusive. Use the first derivative test.
   - If $f'' > 0$, the graph is concave upward and the critical value is a relative minimum.
   - If $f'' < 0$, the graph is concave downward and the critical value is a relative maximum.
3) Find the y-coordinate for each critical value. (If *f* is undefined, there is <u>not</u> an extremum.)

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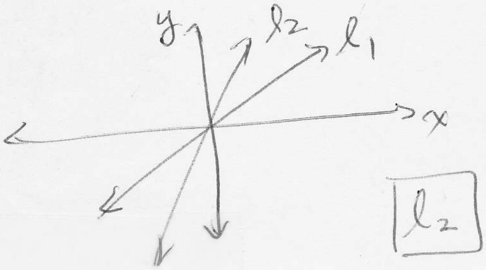
Conceptually

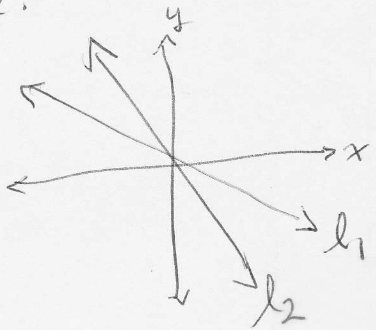
- This section is about the second derivative and what it can tell us about a graph.
- Taking a derivative of a function is finding the rate of change of that function.
- So we're looking at the rate of change of the derivative, meaning the rate of change of <u>slope</u> of the tangent line

EX

① which line has greater slope?



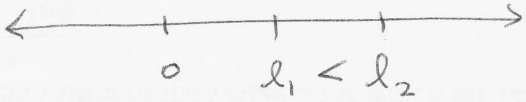
$\boxed{l_2}$



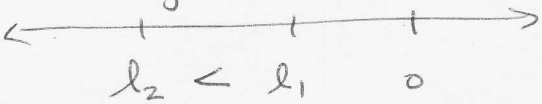
both are neg #s

$\boxed{l_1}$

positive

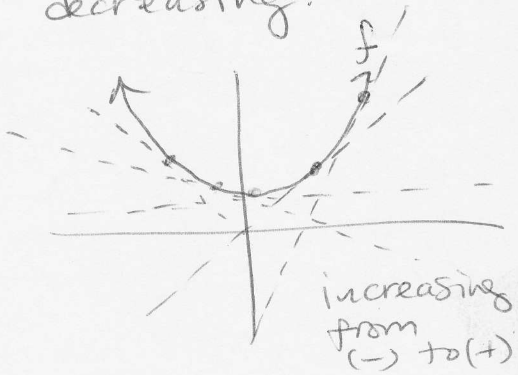


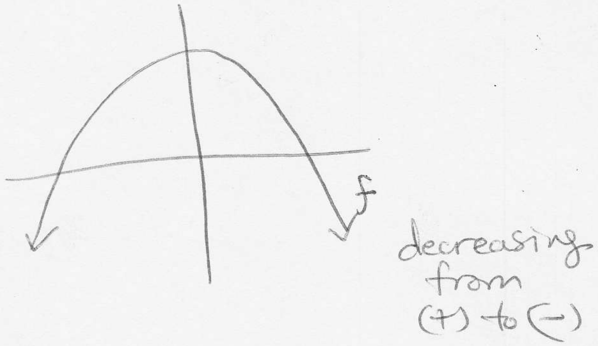
negative



on a number line

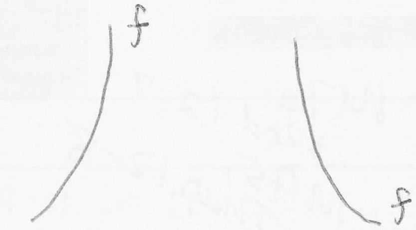
Ex ⑨ Is the slope of the tangent line increasing or decreasing?





When the slope of the tangent line is increasing from left to right, this causes the graph to have a particular shape, called <u>concave up</u>.

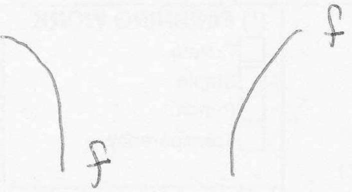
Concave up looks like this:

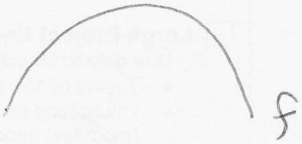


[The graph is some part of a smile. "upper"!]

The rate of change of f' is increasing$_{L\to R}$ $\Leftrightarrow$ f'' is positive

When the slope of the tangent line is decreasing from left to right, this causes the graph to be <u>concave down</u>:





[The graph is some part of a frown. "downer".]

The rate of change of f' is decreasing$_{L\to R}$ $\Leftrightarrow$ f'' is negative.

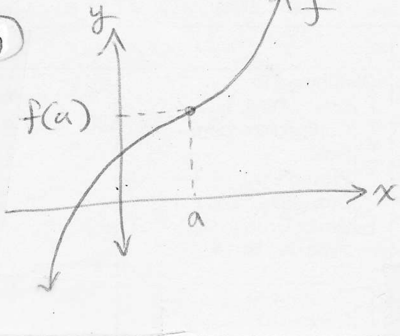
<u>Defn</u>: If $f''(x) > 0$ $\forall x$ on an open interval then f is <u>concave up</u> on that interval

<u>Defn</u> If $f''(x) < 0$ $\forall x$ on an open interval then f is <u>concave down</u> on that interval

A point on the graph where concavity changes from one type to the other is called a <u>point of inflection</u> or <u>inflection point</u>.

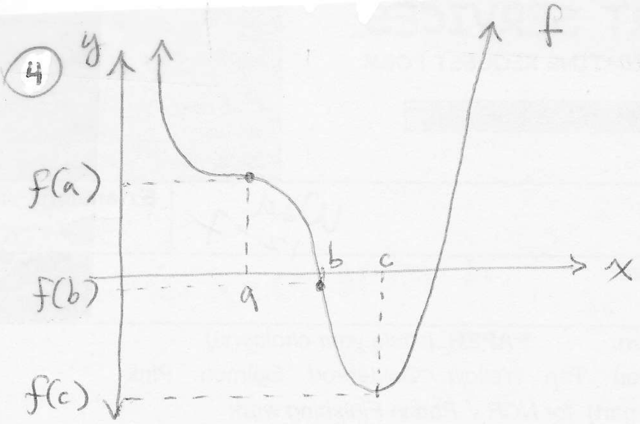
<u>CAUTION</u>: To be a point of inflection, the point must be on the graph. That is, the function must be defined at the point.

③



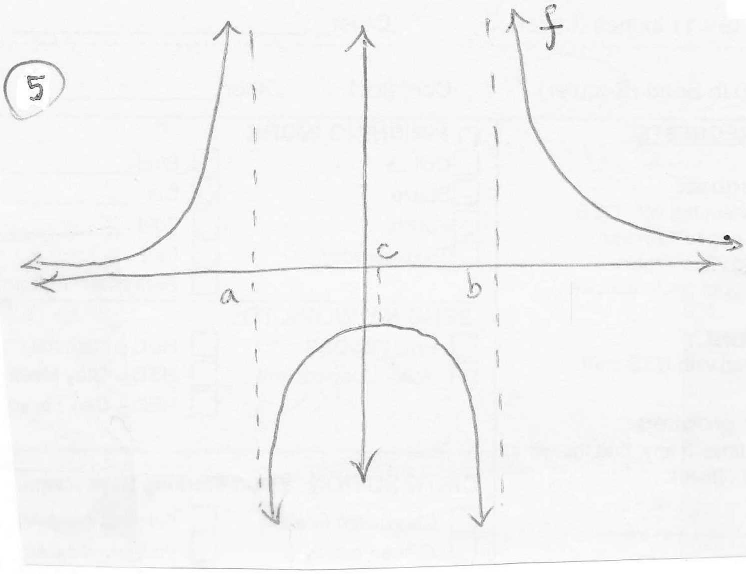
f is concave up (a, ∞)
f is concave down $(-\infty, a)$
$(a, f(a))$ is an inflection point

④



f is concave up $(-\infty, a) \cup (b, \infty)$
f is concave down (a, b).
$(a, f(a))$ and $(b, f(b))$ are inflection points.

⑤



f is concave up $(-\infty, a) \cup (b, \infty)$
f is concave down (a, b)

f has no inflection points
($f(a)$ and $f(b)$ are not defined)

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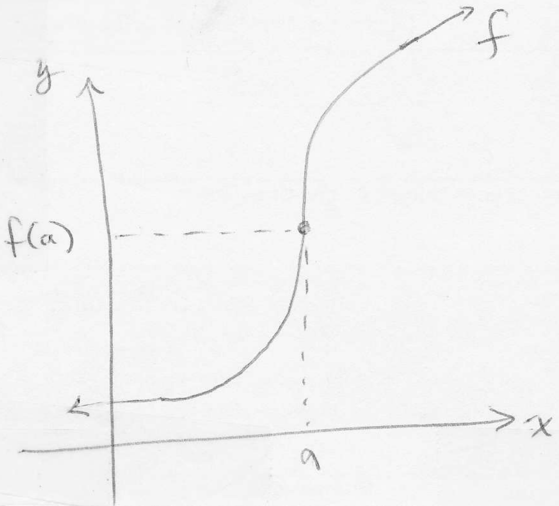
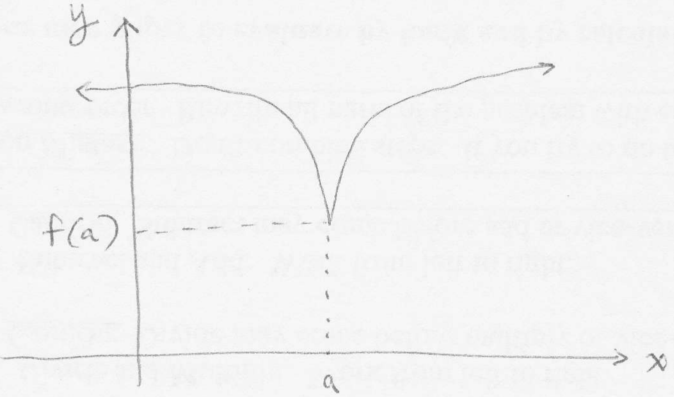
⑥



f is concave up on $(-\infty, a)$
f is concave down on (a, ∞).
$(a, f(a))$ is an inflection pt.

f(a) is defined
even though f'(a) is not defined.

⑦



f is concave down
on $(-\infty, a) \cup (a, \infty)$

$(a, f(a))$ is not an
inflection point

f(a) is defined
f'(a) is not defined.

We can sometimes use concavity to determine if a critical value is a relative extremum.

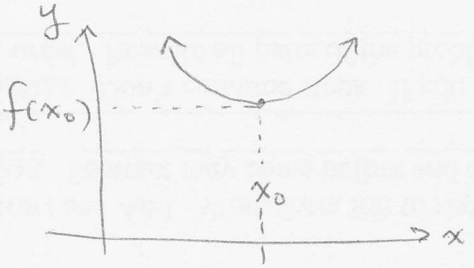
Second Derivative Test

After finding critical values x_0 (where $f'(x_0)=0$ or $f'(x_0)$ is undefined)

consider $f''(x_0)$, the measure of concavity at the critical value x_0.

Four cases:

1) $f''(x_0) > 0$ graph at the critical value x_0 is concave upward



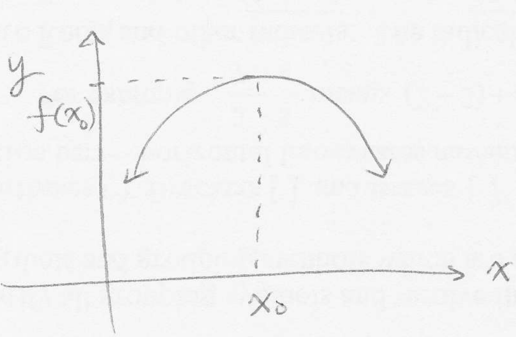
$f(x_0) = y$-coord = min value

$f'(x_0) = 0$ critical value

$f''(x_0) > 0$ concave up.

relative min $f(x_0)$ at $x = x_0$.

2) $f''(x) < 0$ graph at the critical value x_0 is concave downward



$f(x_0) = y$-coord = max value

$f'(x_0) = 0$ critical value

$f''(x_0) < 0$ concave down

These first two cases are <u>conclusive</u> –
the second derivative determines that there <u>is</u> a relative extremum.

The next two cases are <u>inconclusive</u> –

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3) $f''(x_0)=0.$
4) $f''(x_0)$ undefined

If either of these results, cannot conclude anything –

- might be rel. max at x_0
- might be rel. min at x_0
- might be no extremum at x_0.

$\Rightarrow$ Must use first derivative test.

Example of cases 1 & 2

⑧ $f(x)=x^4-13x^2+36 = (x^2-9)(x^2-4)$

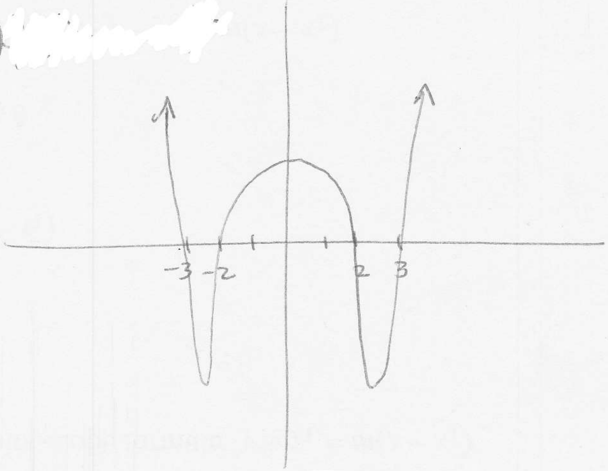
$f'(x)=4x^3-26x+0$

$2x(2x^2-13)=0$

$x=0, \quad 2x^2=13$

$x^2=\frac{13}{2}$

$x=\pm\sqrt{\frac{13}{2}}$



$f''(x)=12x^2-26$

$f''(0)=12(0)^2-26=-26<0$ ⌒ concave down — rel max at $x_0=0$

$f''\left(+\sqrt{\frac{13}{2}}\right)=12\left(\sqrt{\frac{13}{2}}\right)^2-26$

$=12\cdot\frac{13}{2}-26$ concave up

$=78-26=52>0$ ◡ — rel min at $x_0=\sqrt{\frac{13}{2}}$

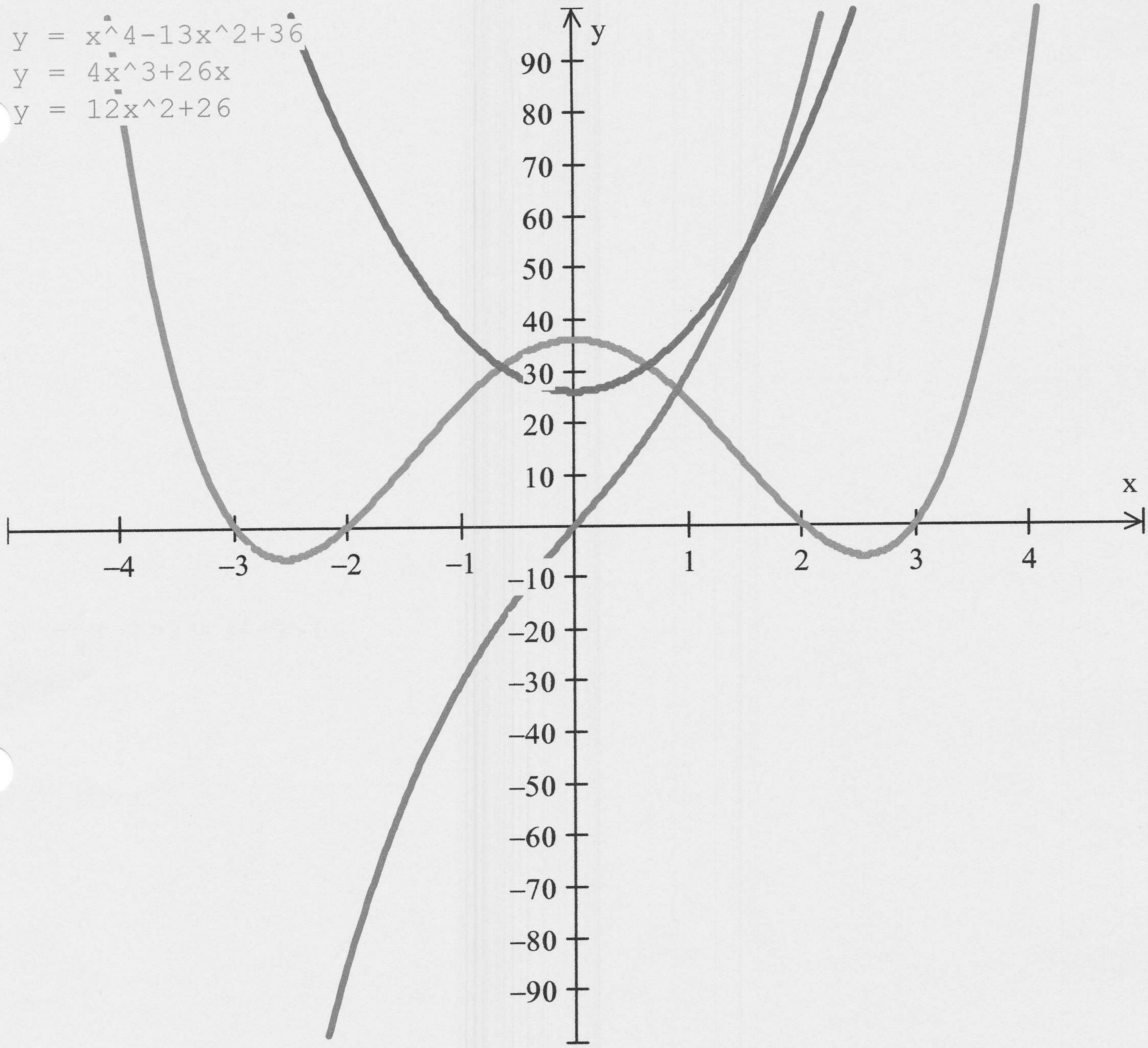
$f''\left(-\sqrt{\frac{13}{2}}\right)=12\left(-\sqrt{\frac{13}{2}}\right)^2-26=52>0$ ◡ concave up — rel min at $x_0=-\sqrt{\frac{13}{2}}$

$f(0)=36$

$f\left(\sqrt{\frac{13}{2}}\right)=\frac{169}{4}-13\cdot\frac{13}{2}+36=\frac{-25}{4}$

$f\left(-\sqrt{\frac{13}{2}}\right)=\frac{-25}{4}$

$$\boxed{\begin{array}{l}\text{rel max } 36 \text{ at } x=0\\ \text{rel min } \frac{-25}{4} \text{ at } x=\sqrt{\frac{13}{2}}\\ \text{rel min } \frac{-25}{4} \text{ at } x=-\sqrt{\frac{13}{2}}\end{array}}$$


y = x^4-13x^2+36
y = 4x^3+26x
y = 12x^2+26
y
x
90
80
70
60
50
40
30
20
10
–10
–20
–30
–40
–50
–60
–70
–80
–90
–4
–3
–2
–1
1
2
3
4

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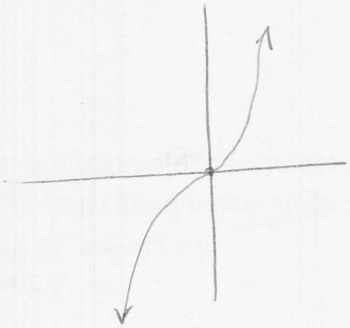
Examples of case 3) $f''(x_0)=0$ inconclusive

⑨ $f(x)=x^3$

$f'(x)=3x^2$

$3x^2=0$

$x=0$ critical value



graph suggests $x_0=0$ is neither max nor min.

1st derivative test proves it's neither max nor min.

$f''(x)=6x$

$f''(0)=6(0)=0$ inconclusive.

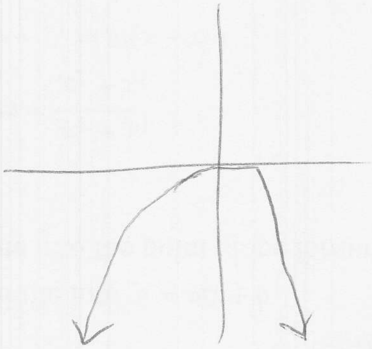
$\boxed{\text{no rel extrema}}$

⑩ $f(x)=-x^4$

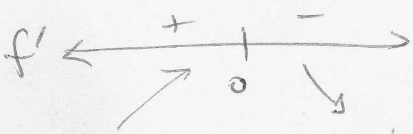
$f'(x)=-4x^3$

$-4x^3=0$

$x=0$ critical value



graph suggests $x_0=0$ is a rel max



1st derivative test proves it's a rel. max.

$f''(x)=-12x^2$

$f''(0)=-12(0)^2=0$ inconclusive

$f(0)=0$

$\boxed{\text{rel max } 0 \text{ at } x=0}$

M250 Determine intervals of concavity, inflection points and relative extrema using the second derivative test.

MUST * (11) $f(x) = x\sqrt{9-x}$

$f(x) = x(9-x)^{\frac{1}{2}}$

$f'(x) = x \cdot \frac{1}{2}(9-x)^{-\frac{1}{2}}(-1) + (9-x)^{\frac{1}{2}} \cdot 1$

$= -\frac{1}{2}(9-x)^{-\frac{1}{2}}\left[x - 2(9-x)\right]$

$= \frac{-1}{2\sqrt{9-x}}(x - 18 + 2x)$

$= \frac{-(3x-18)}{2\sqrt{9-x}}$

$= \frac{-3(x-6)}{2\sqrt{9-x}}$

$= -\frac{3}{2}(x-6)(9-x)^{-\frac{1}{2}}$

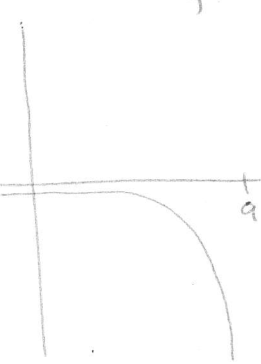
$f''(x) = -\frac{3}{2}\left\{(x-6)\cdot\left(-\frac{1}{2}\right)(9-x)^{-3/2}(-1) + 1\cdot(9-x)^{-\frac{1}{2}}\right\}$

$= -\frac{3}{4}(9-x)^{-3/2}\left\{x - 6 + 2(9-x)\right\}$

$= -\frac{3}{4}(9-x)^{-3/2}\left[x - 6 + 18 - 2x\right]$

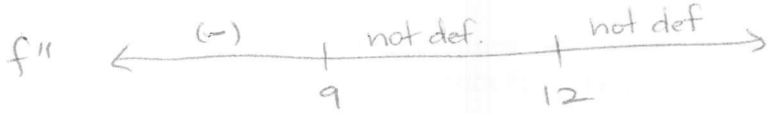
$= -\frac{3}{4}(9-x)^{-3/2}\left[-x + 12\right]$

$= \frac{3}{4}(9-x)^{-3/2}(x-12)$



$f''(x) = 0$ when $x = 12$ except not defined!

f'' undef when $x = 9$



concave down $(-\infty, 9)$

no inflection points

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⑪ con't

$f''(x) = (x^2+1)^{-3/2}\left[-x^2 + x^2 + 1\right]$

$= (x^2+1)^{-3/2}$

$f''(x) = \dfrac{1}{\sqrt{(x^2+1)^3}}$

$f''(x) = 0$ nowhere

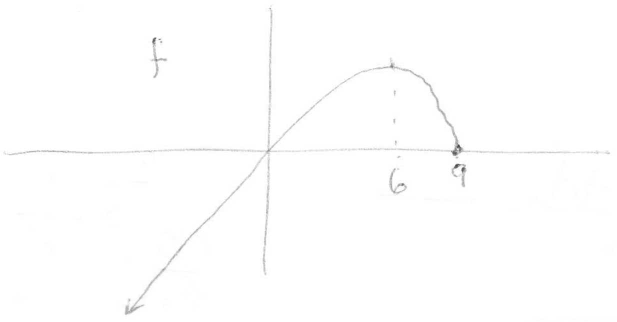
$f''(x)$ under $x^2+1=0$ imaginary

f'' ⟵ (+) ⟶

$\boxed{\text{concave up } (-\infty, \infty)}$

$\boxed{\text{no points of inflection}}$

Consider graph of f



critical values $f'(x) = 0$ $x-6=0$ $\boxed{x=6}$ stationary pt.

$f'(x)$ undef $9-x=0$ $\boxed{x=9}$ not stationary

$f''(6) = \frac{3}{4}(9-6)^{-3/2}(6-12) < 0$ ⌒ concave down, max

$f(6) = 6\sqrt{9-6} = 6\sqrt{3}$

$\boxed{\text{rel max } 6\sqrt{3} \text{ at } x=6}$

$f''(9) = \frac{3}{4}(9-9)^{-3/2}(9-12)$ undefined

2nd derivative test inconclusive

Back to first derivative test

$f(9) = 9\sqrt{9-9} = 0$

$\boxed{\text{rel min } 0 \text{ at } x=9}$

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Examples of case 4) $f''(x_0)$ undef inconclusive

⑫ $f(x) = x^{1/3}$

$f'(x) = \frac{1}{3}x^{-1/3}$

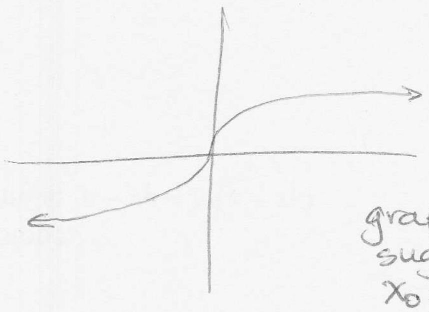
$= \frac{1}{3\sqrt[3]{x}}$ never $f'(x)=0$

undefined @ $x=0$ critical value

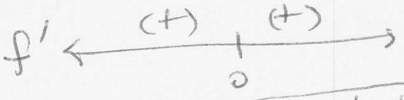
$f''(x) = -\frac{1}{9}x^{-4/3}$

$f''(0) = \frac{1}{9\sqrt[3]{0^4}}$ = undefined inconclusive

$\boxed{\text{no relative extrema}}$



graph suggests $x_0=0$ is neither max nor min.



$\boxed{\text{1st derivative test proves it is neither max nor min.}}$

⑬ $f(x) = x^{4/3}$

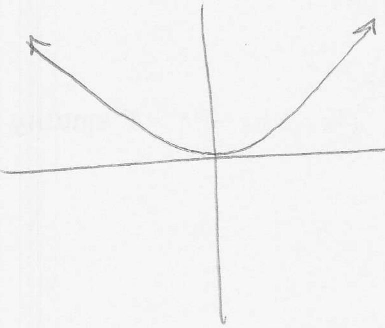
$f'(x) = \frac{4}{3}x^{1/3}$

$f'(x)=0$ when $x_0=0$ critical value

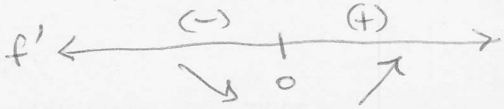
$f''(x) = \frac{4}{9}x^{-2/3}$

$f''(0) = \frac{4}{9\sqrt[3]{0^2}}$ = undefined inconclusive

$f(0)=0$



graph suggests $x_0=0$ is a rel. min.



1st derivative test proves it's a rel. min.

$\boxed{\text{rel. min 0 at } x=0}$

⑭ $f(x) = -x^{2/3}$

$f'(x) = -\frac{2}{3}x^{-1/3}$

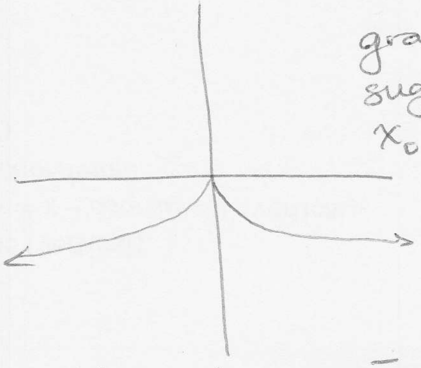
$\frac{-2}{3\sqrt[3]{x}} = 0$ undefined @ $x=0$ critical value

$f''(x) = \frac{2}{9}x^{-4/3}$

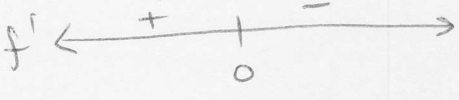
$f''(0) = \frac{2}{9\sqrt[3]{0^4}}$ = undefined inconclusive

$f(0)=0$

$\boxed{\text{rel max 0 at } x=0}$



graph suggests $x_0=0$ is a rel max.



1st derivative test proves it's a rel max.